KOLMOGOROV CONSTANTS IN THE SPECTRA OF ANISOTROPIC TURBULENCE

S. R. Bogdanov

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In [1, 2] a method, based on the scaling hypothesis (SH), is proposed for describing developed isotropic grid turbulence. In this method the perturbations of the velocity field in the energy-containing and inertial intervals are described by universal spectral functions. The dependence on the distance x to the grid is completely determined by only two quantities, which can be termed secular: The average rate of dissipation of energy $\langle\varepsilon\rangle$ and the correlation radius (the integral scale) of turbulence r_c. In particular, for the components of the tensor

$$F_{ij} \equiv (2\pi)^{-3} \int \langle u_i(\mathbf{x})\, u_j(\mathbf{x}+\mathbf{r})\rangle \exp(-i\mathbf{k}\mathbf{r})\, d\mathbf{r} = P_{ij}F \tag{1}$$

this dependence is given by the relation

$$\overline{F}(k, x) = \overline{r}_c^{\beta+2}\varphi(kr_c), \quad \overline{k} \ll 1. \tag{2}$$

Here $P_{ij} = \delta_{ij} - \theta_i\theta_j$; $\theta_i = k_i/k$; $k = |\mathbf{k}|$; $F = F_{ii}$; $\beta \approx 5/3$ is the spectral index; the overbar indicates that the quantity has been made dimensionless with the help of the Kolmogorov length scale $r_d = (\eta^3/\langle\varepsilon\rangle)^{1/4}$ and time scale $t_d = (\eta/\langle\varepsilon\rangle)^{1/2}$; η is the coefficient of kinematic viscosity; and, φ is a universal function. In the inertial interval, where $kr_c \gg 1$, the function φ has the asymptote $(C_1/4\pi)(kr_c)^{-11/3}$, which corresponds to the well-known expression for the spectral energy density $E(k) = C_1\langle\varepsilon\rangle^{2/3}k^{-5/3}$ [3] (C_1 is Kolmogorov's constant).

This approach makes it possible to calculate the dependence of all turbulence parameters on x. In particular, for C_1 we obtain, using the scale dimension $-\mu/2$ of the field ε,

$$C_1 \sim \left[\mathrm{Re}_M\left(\frac{x-x_0}{M}\right)^{1-n}\right]^{\alpha}, \tag{3}$$

where $\mathrm{Re}_M \equiv UM/\eta$; M is the cell size of the grid; $n = 48/(40 - 3\mu) \approx 1.2$ is the damping exponent of the intensity of turbulence; and, $\alpha = 2\mu/(8 - 3\mu)$.

In extending this method to the case of anisotropic turbulence, there first arises the problem of describing the dependence of the spectral tensors on the orientation θ of the wave vector. Correspondingly, additional secular quantities characterizing the anisotropy must be added to the parameters $\langle\varepsilon\rangle$ and r_c. For example, the components of the Reynolds tensor $\langle u_iu_j\rangle$ [4, 5] as well as the tensor obtained from F_{ij} by ingrating over all possible values of θ are used for the secular quantities [6]. In so doing, the parameterization of the spectral tensors is performed directly, but there arises a functional arbitrariness associated with the determination of the form of the scalar functions. This arbitrariness can be eliminated by linearization only if the anisotropy is weak.

This problem can be approached from a different standpoint. There has now been accumulated a large volume of data [7, 8] indicating absence of isotropy not only in the energy-containing, but also in the inertial interval of wave numbers of anisotropic turbulence. At the same time, both the longitudinal and transverse spectra, which are substantially different from the standpoint of their orientational structure, exhibit sections where the "5/3 law" is satisfied. These two facts are compatible only when in the inertial interval the dependences of the spectral functions on k and θ can be factored. Since, however, on the basis of the scaling hypothesis all long-wavelength disturbances can be described in a unified manner, the factorization should also be preserved in the energy-containing interval. As a result, the following relations, which extend the formulas (1) and (2) to the anisotropic case, can be proposed:

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$$F_{ij}(\mathbf{x}, \mathbf{k}) = f_{ij}(\mathbf{x}, \theta)\, \bar{r}_c^{\beta+2} \varphi(kr_c), \quad \bar{k} \ll 1, \tag{4}$$

and analogous relations for the spectral functions of higher order.

Some consequences of Eq. (4) can be checked directly. Thus, for example, with its help it is easy to derive formulas for the Kolmogorov constants C_{ij} and C_1^a appearing in the expressions for the one-dimensional cross and complete spectra of anisotropic turbulence in the inertial interval [8]:

$$C_{ij} = \frac{C_1}{4\pi} \int_0^{2\pi}\int_0^1 \varkappa^{2/3} f_{ij}(\mathbf{x}, \theta)\, d\varkappa\, d\varphi; \quad C_1^a = \frac{C_1}{4\pi} \int_0^{2\pi}\int_0^1 f_{ii}(\mathbf{x}, \theta)\, d\varkappa\, d\varphi. \tag{5}$$

According to Eq. (5), C_{ij} depend on f_{ij}, and hence they are functions of the anisotropy parameters. The form of these functions is most easily specified for the example of axisymmetric turbulence, for which the parameterization of the tensor f_j is performed using only θ and the unit vector $\mathbf{n}$, direct along the flow [9, 10]:

$$f_{ij} = P_{il}P_{jm}(\gamma_1\delta_{lm} + \gamma_2 n_l n_m), \; \gamma_i = \gamma_i(x, \varkappa), \; \varkappa = (\mathbf{n}, \theta).$$

In [10], where axisymmetric turbulence was calculated numerically on the basis of the DIA model, it is shown that to a first approximation (which in any case is "energetically consistent") the dependence of the quantities γ_i on κ can be neglected. On the basis of such an approximation the integrals in the formulas (5) can be calculated directly:

$$C_{11} = \frac{9}{55}\left(\gamma_1 + \frac{12}{17}\gamma_2\right) C_1; \tag{6}$$

$$C_{22} = \frac{12}{55}\left(\gamma_1 + \frac{15}{136}\gamma_2\right) C_1; \tag{7}$$

$$C_1^a = \left(\gamma_1 + \frac{1}{3}\gamma_2\right) C_1. \tag{8}$$

In the isotropic case, when $f_{ij} = P_{ij}$, $\gamma_1 = 1$, and $\gamma_2 = 0$, the well-known relations of [3] for the constants for the longitudinal (C_2) and transverse (C_2') spectra follow from Eqs. (6)-(8): $C_2 = 2C_{11} = 18C_1/55$, $C_2' = 2C_{22} = 24C_1/55$.

The functions γ_i can be expressed in terms of the components of the Reynolds tensor. Indeed, integrating Eq. (4) over all $\mathbf{k}$ and taking into account the fact that the long-wavelength region makes the main contribution to the integral, we obtain

$$\langle u_i u_j \rangle = \alpha \langle \varepsilon \rangle^{2/3} r_c^{2/3} \left(\frac{2}{3}\gamma_1 \delta_{ij} + \gamma_2 \left(\frac{7}{15} n_i n_j + \frac{1}{15}\delta_{ij}\right)\right),$$

where $\alpha = \int_0^\infty y^2 \varphi(y)\, dy$ is the structure constant. From here it follows that

$$\gamma_2/\gamma_1 = 10\left(\langle u_1^2 \rangle - \langle u_2^2 \rangle\right) / \left(8\langle u_2^2 \rangle - \langle u_1^2 \rangle\right), \tag{9}$$

where the index 1 corresponds to the axis oriented along the flow.

With the help of Eq. (9) it is easy to derive from Eqs. (6)-(8) formulas relating the values of the different Kolmogorov constants in the anisotropic case. In particular,

$$\frac{C_2}{C_2'} = \frac{3}{4}\,\frac{1 + 190z/119}{1 - 65z/68}. \tag{10}$$

Here the parameter $z = (\langle u_1^2 \rangle - \langle u_2^2 \rangle)/(\langle u_1^2 \rangle + 2\langle u_2^2 \rangle)$ characterizes the degree of anisotropy. As $z \to 0$ the formula (10) gives the well-known result $C_2 = 3C_2'/4$. It also qualitatively agrees with the data of [11], where the spectra of the substantially nonisotropic grid turbulence were investigated: The values of the constants of the transverse spectrum were found to be lower than in the isotropic case.

Further predictions can be made by assuming that the Kolmogorov constants of the full spectra of isotropic turbulence are the same as those of anisotropic turbulence. This is equivalent to the assumption that the dependence of the total energy $\langle u^2 \rangle/2$ on x is determined only by the quantities $\langle \varepsilon \rangle$ and r_c in the case of anisotropic turbulence also. This assumption, as follows from Eq. (8), leads to the additional relation

$$\gamma_1 + \gamma_2/3 = 1. \quad (11)$$

The formulas (9) and (11) completely specify the dependence of γ_i on z. The relations (6) and (7) can be put into the form

$$\hat{C}_2 \equiv \frac{55}{18}\frac{C_2}{C_1} = 1 + \frac{190z}{119}; \quad \hat{C}_2' \equiv \frac{55}{24}\frac{C_2'}{C_1} = 1 - \frac{65z}{68}. \quad (12)$$

The parameter z varies from −1/2 to 1, and $\hat{C}_2$ and $\hat{C}_2'$ vary over the intervals [24/119, 309/119] and [201/136, 3/68], respectively.

At first glance, such large ranges for the values of the "constants" C are inconsistent with the experimental data. It should be kept in mind, however, that the range of the experimentally achieved values of z is relatively small: from 0 to 0.12. According to Eq. (2) this results in variations of the order of 18% in C_2 and 12% in C_2'. These variations fall within the known spread in the experimental data.

It is important to note that this spread occurs even in flows with approximately the same Reynolds number Re [12]. Therefore, it cannot be explained only with the help of the formula (3). In this respect the results of [13] are instructive. In [13], where the experiments of Kistler and Vrebalovich [11] were "repeated," the same range of values of Re were significantly smaller. The corresponding numerical values $C_2 = 0.65$, $z = 0.12$ and $C_2 = 0.48 \pm 0.06$, $z = 0.02$ are in good agreement with the calculation based on the formula (12).

The dependence of C_2 on the degree of anisotropy is also indirectly confirmed by investigations of geophysical flows, in particular, in [14] it was found, in a study of the atmospheric layer near the ground, that C_2 increases as the distance to the surface decreases.

In conclusion, we note that, strictly speaking, the dependence of the functions γ_i and κ cannot always be neglected, since Kramers' theorem gives additional restrictions on the form of these functions. Using the data of [15], we shall write the restrictions in the form

$$\gamma_1 \geqslant 0, \ \gamma_1 + \gamma_2 \geqslant 0.$$

Substituting into these formulas the explicit expressions for γ_i, we have

$$8\langle u_2^2\rangle \geqslant \langle u_1^2\rangle \geqslant 2\langle u_2^2\rangle/9. \quad (13)$$

The inequality (13) establishes the region of applicability of the obtained results.

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